5th Australasian Congress on Applied Mechanics, ACAM 2007
10-12 December 2007, Brisbane, Australia

A common formula for the combined torsional mesh stiffness of spur gears

Timo Kiekbusch [1] and Ian Howard [2]

[1] Department of Mechanical Engineering, University of Kaiserslautern [a]
[2] Department of Mechanical Engineering, Curtin University of Technology, Perth

Abstract: This paper presents the results of a detailed two-dimensional analysis of the torsional stiffness of several pairs of spur gears in mesh using finite element methods. The model on which this research is based allows the generation of pairs of spur gears in contact with several different parameters and includes an adaptive meshing algorithm for the contact zones.
The FEA results from the various model settings are used to develop a common formula for the combined torsional stiffness of spur gears in mesh. The torsional mesh stiffness of gears in mesh consists of three main components which are the body, teeth and contact stiffnesses. The introduced formula uses these three parts to determine the stiffness for a wide range of gear and gear ratio combinations.

Keywords: gear dynamic modelling, spur gear, finite element modelling, torsional mesh stiffness, contact stiffness.

1 Introduction

Gears are used in many different kinds of rotating machinery and they are often a critical part to the function of the machinery. There have been many attempts in recent years to understand and describe the process of the meshing of spur gears. As the process of meshing is very complex and difficult to describe, the finite element analysis is the method of choice to investigate the underlying relationships.
For the model presented in this paper an FEA model of the complete gear arrangement is used. Recent studies on this topic [1-4] have shown that these models produce the best results in comparison with single-tooth models or partial teeth models. As computer hardware and FEA software advances, working with these rather complex models is now feasible.

2 The finite element gear model

The models used in this research were produced by a fully parametrical APDL [b]-Script which generates the shape of the gears and a relatively coarse mesh of the areas as shown in Figure 1.

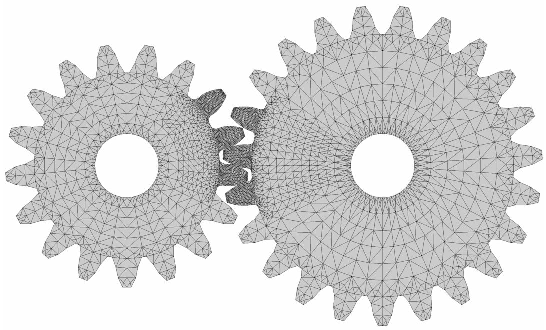
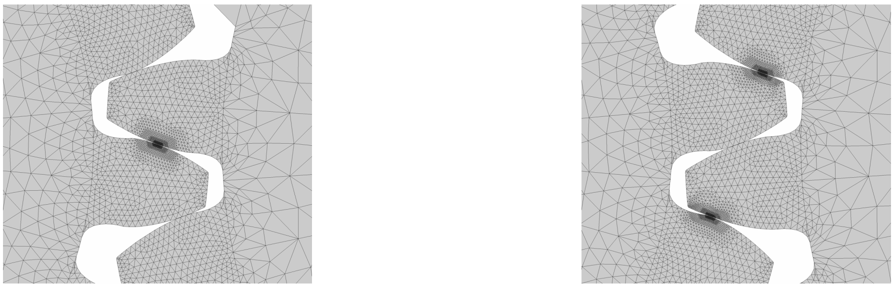


Figure 1 Result from the first APDL-Script; premeshed geometry of pinion and gear

Thereafter the mesh of the model was refined at the contact points using another APDL-Script to keep computing time as low as possible. The result is shown in Figure 2. This script also adds the

[a] In cooperation with Prof. Dr.-Ing. Bernd Sauer, Institute of Machine Elements, Gears and Transmissions, University of Kaiserslautern
[b] ANSYS Parametric Design Language

constraints, contact elements and different torque loads to the model, solves and post processes the job.



2a Remeshed Contact for single contact *2b Remeshed Contacts for double contact*

Figure 2 Adaptive refining of the mesh at the contact point(s)

The constraints used in the model are shown in Figure 3. The hub of the gear was completely constrained from motion; the nodes at the hub of the pinion could only rotate around the centre of the pinion. The torque was applied using a force on every node at the pinion's hub, adding up to the desired torque.

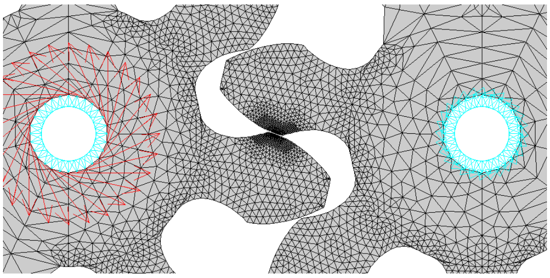


Figure 3 Constraints and loads used in the model (constraints: cyan, loads: red)

The results extracted from the model include the combined torsional mesh stiffness, deformation of gear body, teeth and contact zone for both pinion and gear. These were written to a text file for further processing.
When analysing gears with tooth shape errors or profile correction, an initial gap between the teeth can occur. In order to create a very flexible model which can be used in further studies, the model was built to be able to solve problems including an initial gap between meshing teeth. Typically a static FE model cannot be solved if some parts of the model do not have enough constraints. This can happen, for example, if a contact is not initially closed and in this case the pinion can rotate a small angle without any resistance or stiffness.
To solve models with this rigid body motion, a spring was attached to the pinion to add a small stiffness which prevents any free rotational motion. In order not to falsify the simulation results, this spring is only used for the first simulation step with a very small torque just to get the teeth in contact. In the next simulation step, when applying the required torque, the spring is disabled using the death of elements command.
This approach can handle much bigger gaps (rigid body motion) than the automatic adjustment which tends to be used by default.

3 The combined torsional mesh stiffness

In 2001, Jia introduced a common formula [2], which described the combined torsional mesh stiffness by body and tooth bending stiffness. This formula did not include the effect of the applied torque on the gearing's stiffness. The results of the current research, however, show that there is a significant influence of the torque on the resulting combined torsional mesh stiffness as shown in Figure 4.
Further studies of the deformational behaviour of the gear wheels have shown that the deformation and therefore the stiffness of body and teeth are almost load-independent while the contact

deformation is non-linear, as a Hertzian contact occurs between the teeth of the gears. Figure 5 shows the influence of the applied torque on the stiffnesses of the separate components.
Furthermore, the position and width of the handover region between single and double contact zone are also torque-dependent as has also been shown previously [1].

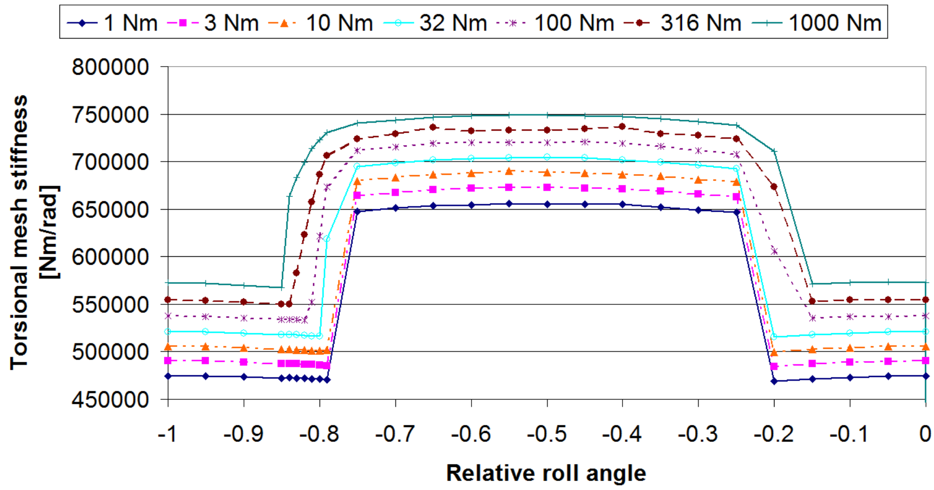


Figure 4 Torsional mesh stiffness for a complete mesh cycle with different torque loads (model with 1:1 gear ratio, 23 teeth, modulus 6mm, steel)

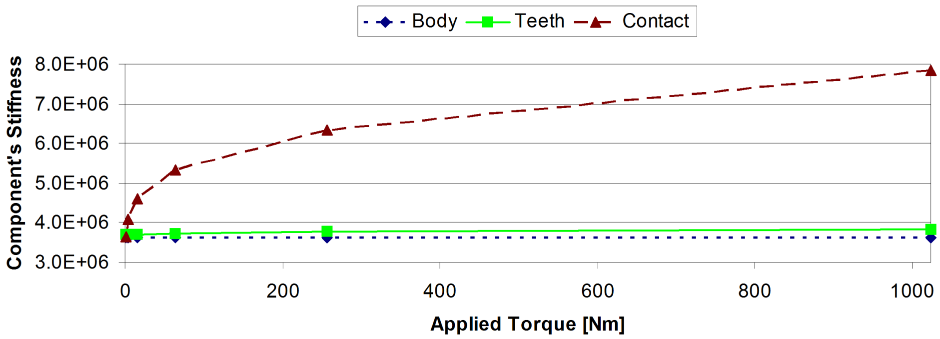


Figure 5 Influence of the applied torque on body, teeth and contact stiffnesses (model with 1:1 gear ratio, 23 teeth, modulus 6mm, steel)

4 Torsional stiffness of a single gear

The three main component stiffnesses (body, teeth and contact) can be considered to act like three springs in a row, which means that the combined stiffness K_i for each pinion and gear is calculated as

$$K_i = \left(K_{B,i}{}^{-1} + K_{T,i}{}^{-1} + K_{C,i}{}^{-1}\right)^{-1} \tag{1}$$

where $K_{B,i}$ is the gear body stiffness, $K_{T,i}$ the teeth stiffness and $K_{C,i}$ the contact stiffness.
These stiffness values are not the actual stiffnesses of the particular component at all, but items which help to develop the common formula for the mesh stiffness. In the following pages the combined gear stiffness K_i is used to calculate the stiffness for the two gears in mesh.
In order to separate the deformations of the body, the teeth and the contact zone, different nodes in the model have been used to read out their displacement. These displacements have been put into relation with the applied torque to obtain the component's stiffness. The nodes which have been chosen to receive the deformation data are placed at the shaft radius, the dedendum radius and at the radius of contact; in each case in the middle of the tooth in contact and in addition one node at the contact point. Figure 6 shows the placement of the selected nodes for a single tooth pair in contact.

The actual deformations of the gear components are gained by looking at the differences between the rotational displacements of the nodes. For instance the deformation of the gear body equals the difference between the displacement of the node at the shaft radius and the one at the dedendum radius.

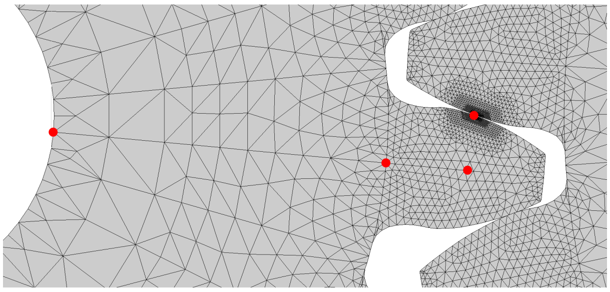


Figure 6 Nodes selected to determine the deformation of gear body, teeth and contact zone (red circles; pinion in the single contact zone)

5 Stiffness for the pinion with a single pair of teeth in contact

In order to determine the equations for the pinion's body, teeth and contact stiffnesses, one needs to think about which gearing parameter affects the particular stiffness. The following indicates those parameters that have been used to describe the stiffnesses beginning with those of a single pair of teeth in contact.
The ranges of the gear model parameters which have been used for this research are shown in Table 1. The applied torque τ was given in Nm and the modulus m in mm.

Table 1 Range of parameters for the models used in this research

Parameter	Min	Max
Number of teeth z	7	50
Modulus m	3	15
Torque τ	0.1	1000
Gear ratio	0.34	2.94

5.1 Stiffness of the gear body

For the gear body stiffness, the body can be considered as a hollow cylinder whose inside is loaded with a torque and a part of its outer area is constrained, which models the particular tooth in contact as shown in Figure 7. The stiffness of this structure was assumed to only depend on the following parameters: shaft radius r_s dedendum radius r_d, face width w and young's modulus E.
From this point a simple model was built to search out the relations between the above mentioned parameters and the body's stiffness. Over one hundred different combinations of parameters were used to come to the following equation:

$$K_{B,P} = c_B \cdot E \cdot w \cdot \ln(r_d - r_s)^{1.6} \cdot r_s^{1.6} \quad (2)$$

where c_B is a coefficient which was found out to be 0.0009555.
The results from this equation reproduce the results from the FEA model within an accuracy of about 5%. The intervals for inner and outer radii which have been used for the studies of the simplified gear body were from $5mm$ to $100mm$ for the inner radius and from $10mm$ to $200mm$ for the outer radius.

5.2 Bending stiffness of the teeth

As the deformation of the teeth basically consists of bending, it can be assumed that the parameters that influence the stiffness of the teeth are the same as those that influence the stiffness of a bending bar. These parameters are: height and width w of the teeth and Young's modulus E. Furthermore, there is an influence of the radius at which the tooth is located, which is a function of both modulus m

and number of teeth z. As the height of the teeth is also a function of the modulus m, the modulus and the teeth number were used to keep the equation as simple as possible:

$$K_{T,P} = c_T \cdot E \cdot w \cdot m^2 \cdot z^{2.2} \quad (3)$$

where c_T is a coefficient which was found out to be 0.000032. The results from this equation are within 7% of the results from the FEA model.

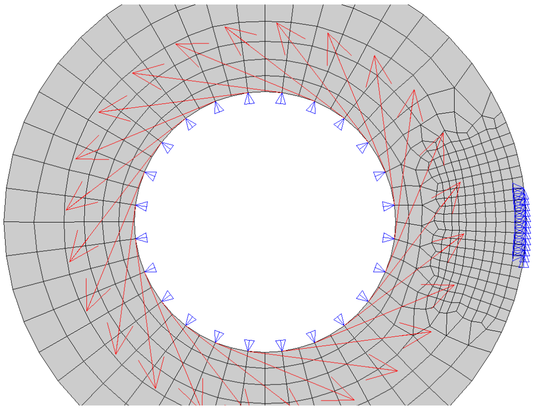


Figure 7 Simplified model of the gear body with applied constraints (blue) and forces (red)

5.3 Contact stiffness

Both stiffnesses mentioned above can be considered to be constant with different torque loads as the deformation is assumed to be linear elastic. The stiffness of contact between the teeth in contact is highly non-linear as it is a Hertzian contact between two curved surfaces. Therefore the torque τ has to be included in the equation used to describe the tooth's stiffness. In addition the following parameters have been found out to have a significant influence: modulus m and number of teeth z, contact radius, Young's modulus E and face width w. The contact stiffness was approximated by

$$K_{C,P} = c_C \cdot E \cdot w \cdot m^{1.85} \cdot z^2 \cdot \tau^{0.105} \quad (4)$$

where the c_C coefficient was equal to 0.000079365. The results for the contact stiffness were found to be within 10% of the results from the FEA model.

5.4 Resulting equation

Using equation (1) the torsional stiffness of the pinion in the single contact zone can be described by:

$$K_{P,\mathrm{single}} = E \cdot w \cdot \left(\left(c_B \cdot \ln\left(r_d - r_s\right)^{1.6} \cdot r_s^{1.6} \right)^{-1} + \left(c_T \cdot m^2 \cdot z^{2.2} \right)^{-1} + \left(c_C \cdot m^{1.85} \cdot z^2 \cdot \tau^{0.105} \right)^{-1} \right)^{-1}. \quad (5)$$

6 Stiffness for the gear with a single pair of teeth in contact

As all stiffness values are related to the pinion, the values of the gear have to be scaled with the gear ratio u squared. This is founded on the fact that both torque and radius of contact of the gear are u-times the respective torque and contact radius of the pinion. So the body, teeth and contact stiffnesses are calculated by:

$$K_{i,G} = K_{i,P} \cdot \left(\frac{z_P}{z_G} \right)^2 \quad (6)$$

with i being the indices B, T and C for body, teeth and contact.

7 Stiffnesses in the double contact zone

The equations for the double contact zone are developed based on the equations mentioned in sections 5 and 6. The influence of the number of teeth in contact on the gear body stiffness was considered to be rather small. The results from the FEA model show that the values are about 10% higher than those from the single contact zone, which is due to the wider area where the force is applied to the body. Hence, the equation from the single contact zone was scaled with the factor f_B which was equal to 1.1:

$$K_{i,B,\text{double}} = f_B \cdot K_{i,B,\text{single}} \tag{7}$$

For the teeth stiffness it was considered here that both teeth pairs share the load equally, which results in the assumption that each tooth's deformation is half the deformation as in the single contact zone. This results in a stiffness twice as high as for the single contact: $K_{i,T,\text{double}} = 2 \cdot K_{i,T,\text{single}}$ (8)

The circumstances concerning the contact's stiffnesses are more complex so that they cannot be described by adding a multiplier. But the analysis of the FEA model results in an equation similar to equation (4) with just a changed exponent for the applied torque:

$$K_{i,C,\text{double}} = c_C \cdot E \cdot w \cdot m^{1.85} \cdot z^2 \cdot \text{t}^{\,0.068} \tag{9}$$

where c_C is 0.00011905.

The resulting equations for the torsional stiffness of pinion and gear are:

$$K_{P,\text{double}} = E \cdot w \cdot \left(\left(f_B \cdot c_B \cdot \ln(r_d - r_s)^{1.6} \cdot r_s^{\,1.6} \right)^{-1} + \left(2 \cdot c_T \cdot m^2 \cdot z^{2.2} \right)^{-1} + \left(c_C \cdot m^{1.85} \cdot z^2 \cdot \text{t}^{\,0.068} \right)^{-1} \right)^{-1} \tag{10}$$

$$K_{G,\text{double}} = E \cdot w \cdot \left(\left(f_B \cdot c_B \cdot \ln(r_d - r_s)^{1.6} \cdot r_s^{\,1.6} \right)^{-1} + \left(2 \cdot c_T \cdot m^2 \cdot z^{2.2} \right)^{-1} + \left(c_C \cdot m^{1.85} \cdot z^2 \cdot \text{t}^{\,0.068} \right)^{-1} \right)^{-1} \cdot \left(\frac{z_P}{z_G} \right)^2 \tag{11}$$

8 The formula for the combined torsional mesh stiffness

As mentioned above the stiffness for the single gear is calculated by assuming all three springs in a row, so that the stiffness for each gear (K_P, K_G) can be calculated. From both of these values the combined mesh stiffness K_m can be derived by considering the two gears as two springs in series:

$$K_m = \left(K_P^{-1} + K_G^{-1} \right)^{-1} \tag{12}$$

This procedure was followed for both the single and double contact zones resulting in the combined torsional mesh stiffness for both zones as shown in Figure 8 which compares the results from the formula developed above and the FEA model.

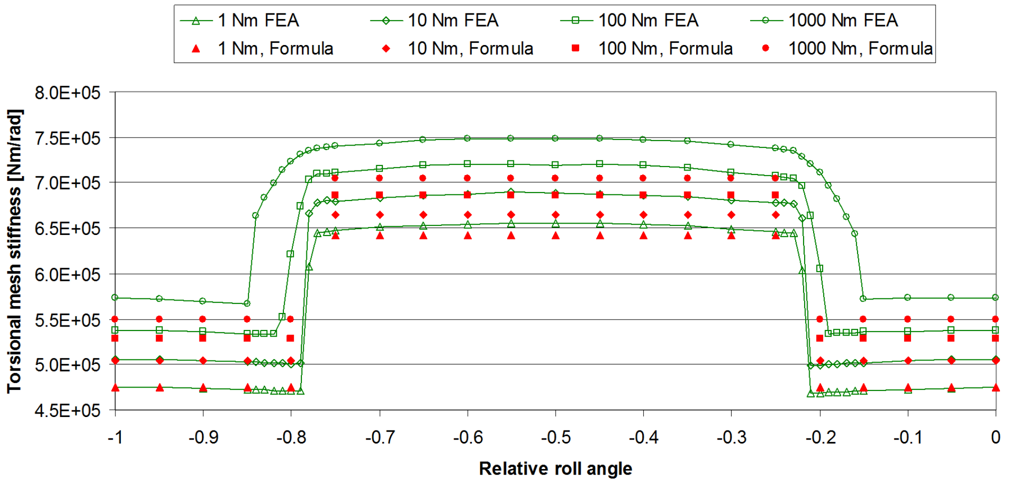


Figure 8. Results for the combined torsional mesh stiffness from the FEA model and from the formula discussed in this paper (model with 1:1 gear ratio, 23 teeth, modulus 6mm, steel)

It is clear that the results from the formula are within 10% from the FEA model's results for the whole range of applied torque. This accuracy seems to be good enough for most of the cases where the formula could be used as there are many other different factors which influence the value of the mesh stiffness, like the shaft to collar connection and the lubrication of the gears.

Figure 8 also shows that the formula does not include the possibility to determine the position and width of the hand-over region, though this will be investigated further.

9 Conclusion

The analysis of numerous different gear models led to a common formula which describes the stiffness of the gears in mesh with the input parameters: number of teeth, modulus, torque, Young's modulus, face width and shaft radii for a whole mesh cycle. The results describe the combined stiffness using different stiffness values for each of the single and the double contact zone. The formula approximation gives results of the FE analysis, which show that there is negligible change of stiffness within each region.

Further studies concerning the hand-over region have to be made to develop an expression for the width of the single and double contact zone and their transition.

References

[1] J. Wang and I. Howard. The torsional stiffness of involute spur gears. In *Proc. Instn Mech. Engrs, Part C: J. Mechanical Engineering Science*, pages 131-142, 2004.

[2] S. Jia, I. Howard, and J. Wang. A common formula for spur gear mesh stiffness. In *Proceedings of the JSME International Conference on Motion and Power Transmissions*, pages 1-4, November 2001.

[3] J. Wang and I. Howard. Finite element analysis of high contact ratio spur gears in mesh. *ASME Journal of Tribology*, Vol. 127, 2005, pages 469 – 483.

[4] S. Jia and I. Howard, Comparison of localised spalling and crack damage from dynamic modelling of spur gear vibrations. *Mechanical Systems and Signal Processing*, Vol (20), 2006, pages 332-349.